The Geometry of a Lamella Roof

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Introduction

The purpose of this monograph is to describe the exact geometry of a lamella roof. Figure 1 shows an example, the ceiling of a workshop I built some years ago. A lamella roof is composed of many small (1m–2m), identical lamellas; these are frequently stock lumber—2×8s, for example. This makes it attractive for the amateur builder. Lamella roofs are strikingly beautiful, and quite easy to build—once you have figured out the geometry.

Basic engineering considerations dictate some of the geometry.[1] The arches should intersect at an angle of roughly 38° to 42°, and, obviously, the lamellas have to be strong enough to carry the load placed on them. The more that the roof is curved the higher the sideways wind load (if the structure supports a roof rather than a ceiling). A typical roof height is between 1/5 and 1/8 of the width of the space it spans. Anything less than four lamellas per arch is a bit crude-looking. The spacing of the arches, dictated by the geometry, cannot be so large that the loads on them get too large. So the general appearance of a lamella roof is relatively constrained.

An engineering analysis is outside the scope of this paper. Some have argued, however, that one reason for the lack of enthusiasm for lamella structures in the last fifty years or so has been the well-advertised failure of large lamella structures with an asymmetric load (e.g., wind, plus snow build-up). This, plus the difficulty of understanding how a lamella structure reacts to failure of one of its elements, such as a weakened lamella, should give anyone pause. But modern engineering software is equal to the task of figuring these kinds of issues out.

Formally the strength of a lamella structure is provided by intersecting, connected, helical arches. For shallow arches, ones without much curvature, the helical nature of the arches is hard to recognize, but Figure 2—a diagram of what a lamella roof would look like if it were continued into a complete

[1] See references for these.

Figure 1: A lamella ceiling

cylinder encircling a space—demonstrates the true nature of the lamella roof.

Figure 3 shows what lamellas looks like. Because the roof is made up of intersecting arches, a "left" and a "right," say, there are, correspondingly, two types of lamella, depending on which type of arch the lamella is part of. The two type are mirror images of each other. The upper surface is curved to conform with the curvature of the roof. The ends are bevelled so that they intersect properly with two lamellas on arches going in the other direction. This can be seen in Figure 4, which shows the under-side of a roof, how the bevels work, and how, if you are going to bolt the lamellas together, the bolt is positioned. (This is just one of several options for joining the lamellas, but you need an engineering analysis to choose an appropriate one.)

Design decisions

There are really only three design decisions for a lamella roof, in terms of its basic geometry.

- The first is the skew-angle ϕ, which is the angle between the direction of the arches and a transverse plane. You can see from the last figure that the diamond pattern has a basic angle of 2ϕ. In this picture the skew-angle ϕ is $22.5°$, and the lamellas intersect at $45°$.[2]

[2]The earlier comment that ϕ should be between $38°$ and $42°$ is an engineering con-

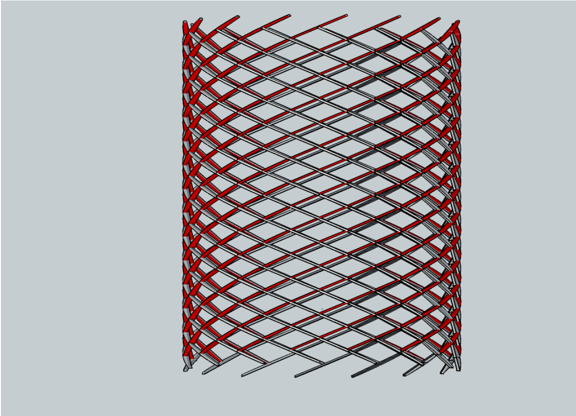

Figure 2: Lamella roof as helical structure

Figure 3: Lamella examples

Figure 4: Characteristic diamond pattern of a lamella roof

The spacing between intersections in a longitudinal direction is just $d = l \sin\phi$. It is a good idea if the lamellas intersect at the walls at the end of a space, as well as at the sides. Since the spacing of intersections (nodes) depends on the choice of ϕ, it is possible to choose ϕ accordingly; this is discussed in the section below on Fabrication Issues.

- The second design decision is the height of an arch, which, together with the span dictates the curvature of the roof. From simple trigonometry (see Figure 5) we know that a section taken across the roof is circular, so if the height is h, and the span is S, the radius of the roof, r is given by:[3]

$$(r-h)^2 + \left(\frac{S}{2}\right)^2 = r^2$$

$$r = \frac{4h^2 + S^2}{8h}$$

- The third design decision is the number of lamellas, n. This is an aesthetic and engineering issue. An interesting roof will have $n \geq 4$, and, the fewer lamellas there are per arch the less the advantage this kind of structure has over a conventional truss.

With ϕ, r, and n determined, the bolt-bolt lamella length l can be calculated. Each lamella, viewed in section, has a length $l' = l \cos\phi$. At the centre of curvature of the roof (in a transverse section of the space) this chord subtends an angle 2θ, where $\theta = \arcsin l'/2r$. n lamellas then subtend an angle of $2n\theta$, and:

$$r \sin n\theta = \frac{S}{2}$$

$$\theta = \frac{\arcsin S/2r}{n}$$

(In the event that the height of the roof is greater than half the span, that is, the diameter of the roof is greater than the span, and the roof subtends more than 180°, $\theta = (\Pi - \arcsin S/2r)/n$.

straint. In the case of the roof in this picture there was no load on it, since it was an interior false ceiling.

[3]This is the radius taken at the centre of a bolt joining two arches. The actual surface of the roof will be further out. This is discussed below.

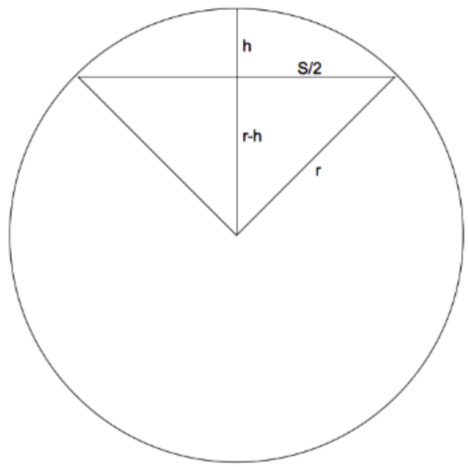

Figure 5: Arch height, roof span, and radius

Since $l' = l\cos\phi$ and $l' = 2r\sin\theta$, we have:

$$l = \frac{2r\sin\theta}{\cos\phi}$$

We can also work out the angle $2\theta'$ that a lamella subtends at the centre of curvature of the roof, in a section taken in the vertical plane of the lamella. From Figure 6 we see that the vertical element can be expressed as either $l/2\tan\theta'$ or as $l\cos\phi/2\tan\theta$, hence:

$$\tan\theta' = \tan\theta/\cos\phi.$$

$$\theta' = \arctan\left(\frac{\tan\theta}{\cos\phi}\right)$$

Basic geometry

The helical arch

In order to understand the geometry of the roof as a whole we need to consider how a helix works. A point in a helix is defined by a rotational angle

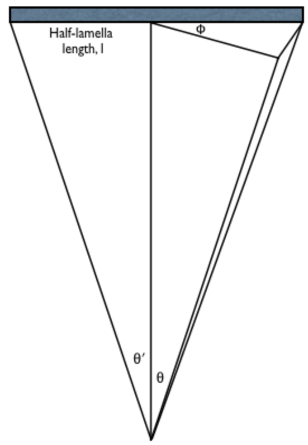

Figure 6: Angle θ' the lamella subtends in its own plane

and a longitudinal distance (all points lying on the surface of a cylinder), such that the distance is some constant times the angle. As the rotational angle increases the helix winds around the cylinder at a constant angle relative to its axis. A screw thread is the best-known example. The lamellas in an arch approximate this behaviour. To be precise, each end of a lamella is located on a helix. The "next" lamella in each arch can be derived from another lamella by rotating the lamella through an angle 2θ, where the axis of rotation is the centre of curvature of the roof, and moving it longitudinally an amount $l \sin \phi$. It's that simple. Figure 7 illustrates the process. The arch in the opposite direction is built up exactly the same way, but it is rotated by an amount θ relative to the first arch. This results in the join of two lamellas in one arch being exactly above the centre of a lamella in an opposing arch.

Now we replicate the arches longitudinally, by the same inter-node distance of $l \sin \phi$, lining them up in the characteristic diamond pattern. In a roof made up of left arches with n lamellas each, all the lamellas in these arches are aligned in n planes, rotated 2θ relative to each other. The arches composed of right lamellas will each have $n - 1$ complete lamellas, and 2 half-lamellas, aligned in $n + 1$ planes. The arches are in fact *woven*, that is going alternately under and over each other. Figure 8 shows how this works.

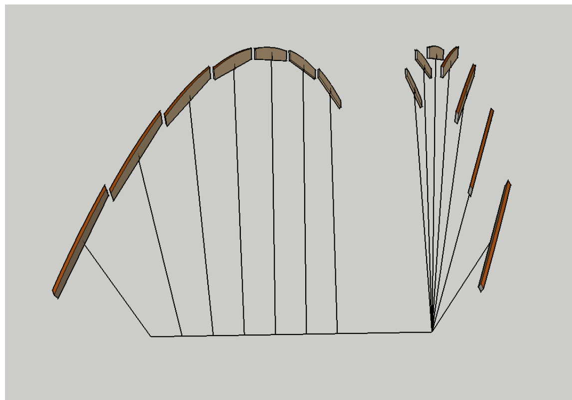

Figure 7: How an arch is built

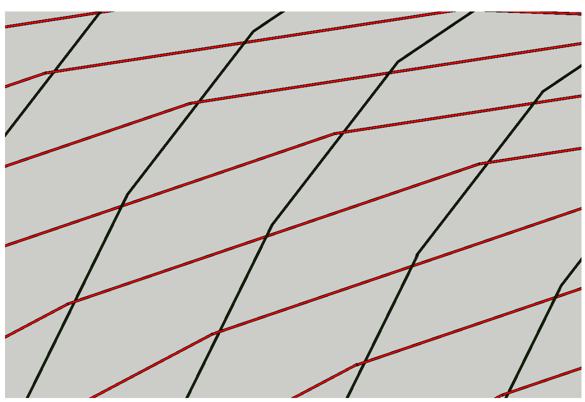

Figure 8: The woven nature of the roof

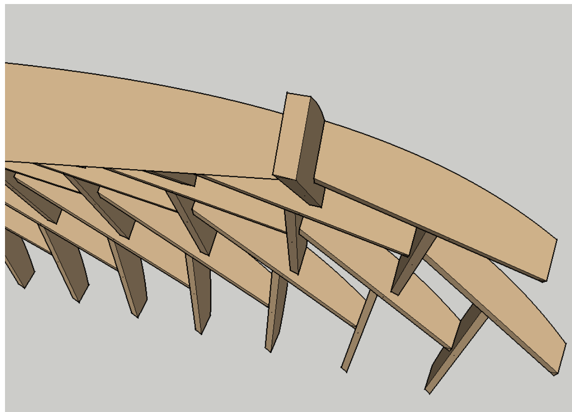

Figure 9: 3-D view of lamella intersection

The lamella

This deals with the fundamental geometry of the roof. Now we turn to the geometry of the lamellas themselves. Three factors have to be considered: first, the orientation of the lamellas relative to each other—how they intersect; second, the thickness of the lamella; and third, the width of the lamella, related partly to the engineering requirements for lamella strength, and partly to the curvature of the upper surface of the lamella.

Intersection of lamellas

The ends of each lamella are bevelled to fit exactly against lamellas in opposing arches. It's a relatively simple exercise in solid geometry to figure the bevel angles out. Because the lamella structure repeats itself exactly—it has rotational and translational symmetry—it is only necessary to model a single intersection in the structure. Consider the plan view of an intersection, above one of the lamellas which is horizontal. At the centre of the lamella two other lamellas, oriented at 2ϕ, are bolted to it. But they are not in the same plane—the two lamellas that touch this central lamella are tilted downwards at an angle θ viewed in section. (This is illustrated in Figure 9.)

The way the analysis proceeds is to consider the normals to the intersecting planes, and from their cross-product establish the lines of intersection. After this it is straightforward to find the bevel angles. Suppose we label the "vertical" planes of two intersecting lamellas $\boldsymbol{A}$ and $\boldsymbol{B}$. Figure 10 is a diagram showing the relationship between the planes $\boldsymbol{A}$ and $\boldsymbol{B}$, defined by

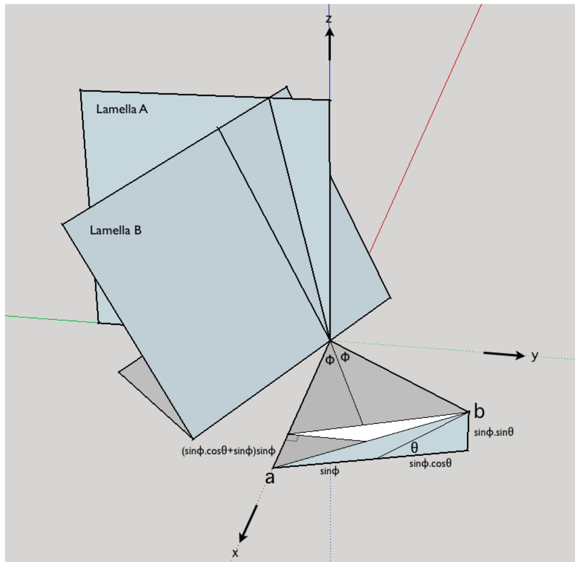

Figure 10: Intersection of lamella planes

the unit normals $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{b} = \begin{bmatrix} 1 - (\sin\phi\cos\theta + \sin\phi)\sin\phi \\ (1+\cos\theta)\sin\phi\cos\phi \\ \sin\phi\sin\theta \end{bmatrix}$$

By definition the line of intersection $\boldsymbol{c}$ of the two planes lies in both planes, and therefore must be perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. It is given by the cross-product of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{c} = \begin{bmatrix} 0 \\ \sin\phi\sin\theta \\ (1+\cos\theta)\sin\phi\cos\phi \end{bmatrix}$$

The "vertical" bevel angle is just $\arctan(c_y/c_z)$. Note that this is:

$$\chi = \arctan\left(\frac{\sin\phi\sin\theta}{(1+\cos\theta)\sin\phi\cos\phi}\right)$$

$$\chi = \arctan\left(\frac{\tan\theta}{(1/\cos\theta+1)\cos\phi}\right)$$

But:

$$\frac{\tan\theta}{\cos\phi} = \tan\theta'$$

so as θ gets smaller,

$$\chi \approx \arctan\left(\frac{\tan\theta'}{2}\right) \approx \left(\frac{\theta'}{2}\right)$$

A similar calculation gives the bevel angle in the "horizontal" direction. In this case the "horizontal" plane $\boldsymbol{F}$ is defined by its normal $\boldsymbol{f}$:

$$\boldsymbol{f} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The cross-product of $\boldsymbol{b}$ and $\boldsymbol{f}$ is:

$$\boldsymbol{g} = \begin{bmatrix} (\sin\phi + \sin\phi\cos\theta)\cos\phi \\ 1 - \sin\phi(\sin\phi\cos\theta + \sin\phi \\ 0 \end{bmatrix}$$

The horizontal bevel angle ψ is then:

$$\psi = \arctan\left(\frac{\sin\phi\cos\phi(1+\cos\theta)}{1 - sin^2\phi(1+\cos\theta)}\right).$$

When θ is small, then $1 + \cos\theta \approx 2$, and we have:

$$\psi \approx \arctan\left(\frac{2\sin\phi\cos\phi}{1 - 2sin^2\phi}\right) = \arctan\frac{\sin 2\phi}{cos2\phi}$$

$$\psi \approx 2\phi$$

Since θ is typically less than $10°$, $\cos\theta > 0.98$, the approximations for χ and ψ are very good. The alert reader (you) may wonder why the vertical bevel angle is $\theta/2$, rather than the θ one might expect intuitively. The answer lies in the helical twist of the lamellas. Each lamella is a planar approximation to a twisted ribbon, so either the lamella has to twist (preserving the bevel angles at 2ϕ and θ), or the bevel angles, particularly the bevel in the "vertical" plane, have to accommodate the twist. A planar lamella leans in towards the lamella it touches, at each end, and it is this lean that reduces the bevel angle to $\approx \theta/2$.

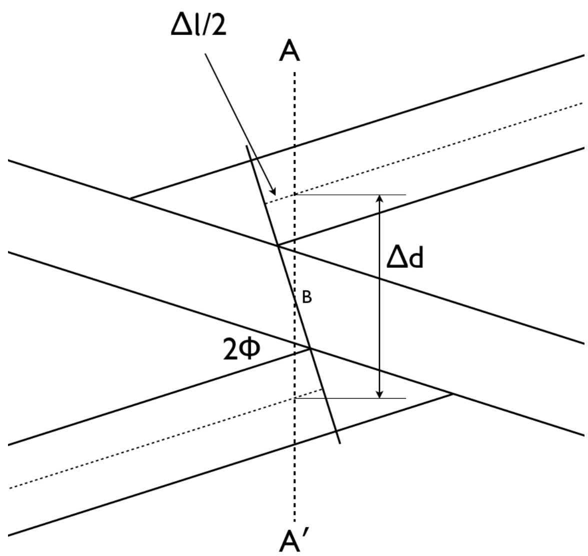

Figure 11: Plan of intersection

The effect of lamella thickness

We now introduce the complication that the lamellas have some thickness, t. The effect of this on the geometry can be seen from a top-down schematic of the intersection of two arches (see Figure 11). Where one lamella meets another on an intersecting arch its end will be bevelled to lie flat against the other one. A variety of ways are available to tie the lamellas together—the easiest is to bolt through them at a 90° angle to the outer lamellas. The thickness of the bolt will affect the geometry of the roof, but, to start with, assume, as represented in the diagram, a zero-thickness "bolt." If the lamella had no thickness the ends of the two lamellas L_1 and L_2 would meet at point B on the line A–A'. The effect of lamella thickness is to move the lamellas apart, while maintaining their original ends on the line A–A'. Now, instead of meeting, the original ends of the lamella are separated by a distance Δd, an increase in the inter-node distance, previously $l \sin \phi$. And, the lamellas will have to be longer, as measured bolt-to-bolt, by an amount Δl. From the geometry we see that:

$$\Delta l = t \tan \phi (1 + \sec 2\phi)$$

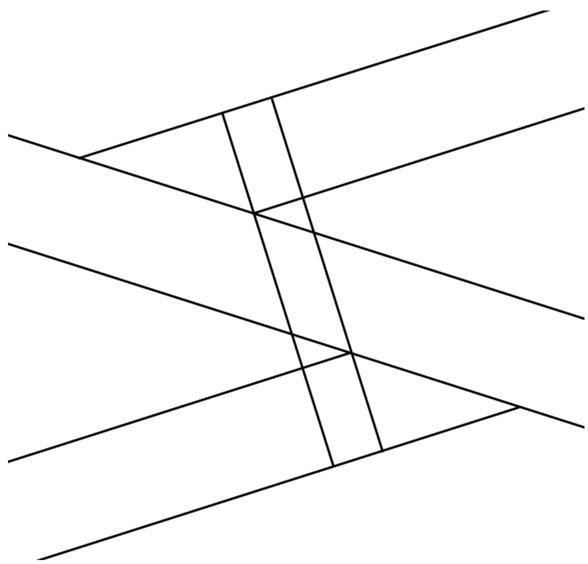

Figure 12: Effect of bolt diameter

$$\Delta d = t(1 + \sec 2\phi) \sec \phi$$

In reality the bolt will have some diameter b, and it is best located such that it passes through the full thickness of the lamellas, as shown in Figure 12. This has the effect of moving the lamellas even further apart, for which their length has to be adjusted. The overall adjustments for lamella thickness and bolt diameter are then:

$$\Delta l = t \tan \phi(1 + \sec 2\phi) + b(1 + \tan 2\phi \tan \phi)$$

$$\Delta d = t(1 + \sec 2\phi) \sec \phi + b \tan 2\phi / cos\phi$$

.

The angular orientations of the lamellas are unchanged.

The blank from which a lamella is cut will also have to reflect the incremental length required to accommodate the horizontal bevel ψ. This is obviously $2t/\tan \psi \approx 2t/\tan 2\phi$.

The built-up roof now looks like Figure 13, with “horizontal” strips replacing the earlier rods representing lamellas.

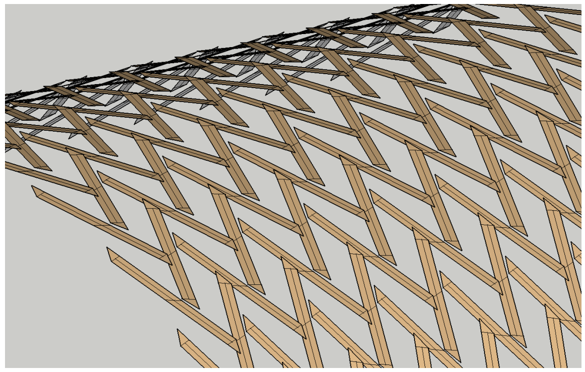

Figure 13: Effect of lamella thickness

The effect of lamella width

We have now modelled the lamellas as interconnected strips in helical arches. The lamellas will also have width dependent on the engineering requirements and the curvature of the roof. The profile of a lamella on its inner surface is arbitrary, but the profile on the outer surface has to conform to a cylindrical surface with an axis at the centre of curvature of the roof. How far this surface is from the points of interconnection of the lamellas is, again, an engineering issue. Earlier we imagined a roof with radius r, this radius being the distance from the centre of the roof to a point of intersection of lamellas. Clearly the actual interior surface of the roof will be circular in cross-section, concentric with the circle describing the intersection points of the lamellas. The outer profile of a lamella is defined by the intersection of a roof cylinder with radius r' and the sides of the lamella. Figure 14 shows how this works. A plane at a skew-angle ϕ to a section across a cylinder will have an elliptical shape, where the minor axis is just r', and the major axis is $r'/\cos\phi$. The lamella blank will have to accommodate this curvature. If the resultant width is W, then the incremental length attributable to the width is $2W\tan\chi$.

Fabrication issues

Where the roof meets the side walls there has to be some way of fixing the lamellas to the wall. A variety of ways is possible, from the use of steel brackets to the more conventional cutting of bird's-mouths in the lamellas.

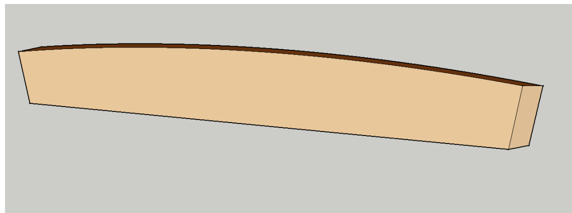

Figure 14: Curved upper surface of lamella

Here we describe the geometry of the edge of the roof.

First, recognize that the planning assumption is that the end of a lamella taken at its unadjusted length is the point that is incident on the wall at a distance $S/2$ from the middle of the space. Note that if there are, say, 6 lamellas to a left arch, then there will be 5 complete lamellas in each right arch, and two half-lamellas. The exact form of these half-lamellas depends on how the roof is fixed to the wall (they aren't just chopped off half way along their length). Figure 15 shows how the edge of the roof might look. In this case the half-lamellas have the bird's-mouth cut. Most frequently there will be blocking of some kind to support the lamellas.

Consider the geometry of a whole (left) lamella at the edge of the roof. First, we must figure out what angle it is lying at. A normal to the lower edge of the lamella is positioned at an angle $(n-1)\theta$ to the vertical. (Remember that each lamella subtends an angle 2θ when viewed in section.) The lamella itself is skewed at an angle ϕ. Thus a normal to the face of the left lamella can be drawn as:

$$\begin{bmatrix} \cos\phi \\ \sin\phi\cos(n-1)\,\theta \\ \sin\phi\sin(n-1)\,\theta \end{bmatrix}$$

A normal to the face of a half-lamella is similar, except that the relevant vertical angle is now $n\theta$:

$$\begin{bmatrix} \cos\phi \\ \sin\phi\cos n\theta \\ \sin\phi\sin n\theta \end{bmatrix}$$

A similar calculation to that described earlier for determining the angles of intersection of lamellas with each other gives the cut angles. Note that the lower edges of the lamellas meet both the horizontal and vertical wall surfaces with a horizontal line of intersection, conveniently at an angle ϕ relative to the edge of the lamella, so it is easy to transfer the cut from one

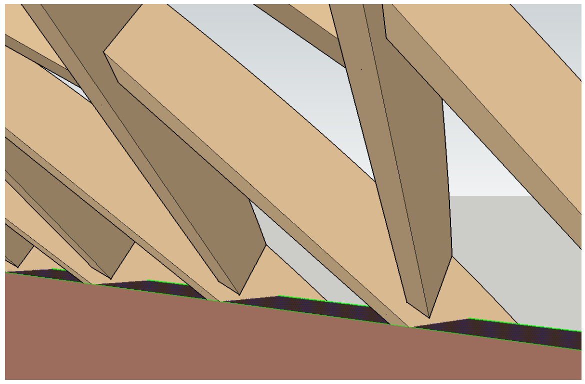

Figure 15: At the edge of the roof

side of the lamella to another.

Where the lamellas meet the gable walls there is a similar issue of how to fix them to the walls, and support them. The geometry of all the intersections is the same (i.e., it doesn't depend on where on the curve of the roof a lamella is. But the same issue of half-lamellas arises. Figure 16 shows how a gable end might look. The bevel angles on the lamellas are just ϕ on the lower face of a lamella, and $90°$ on the side face, for both left and right lamellas.

One can generate a set of equations that translate the design parameters into the resulting distance between nodes, and hence what the length of the space would be if the end-nodes were exactly on the gable walls. I don't think it's possible to invert these equations to extract the design parameters for a required length, but, trivially, by trial-and-error, one can establish a set of parameters that make this distance exactly equal to the required length. The easiest parameter to change is the height of the roof. This is illustrated in the worked example in the next section.

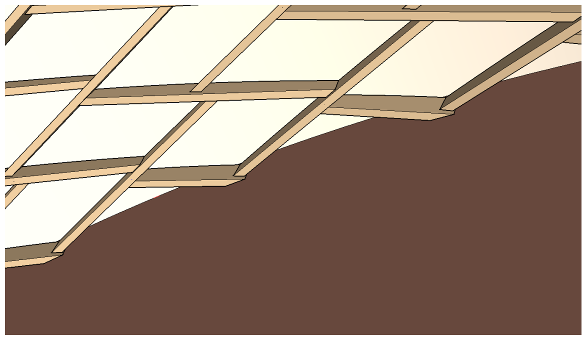

Figure 16: At the gable end of the roof

A worked example

Here are the initial conditions:

$$\begin{aligned}
\text{Span, } S &= 6000\,\text{mm} \\
\text{Height of roof, } h &= S/5 = 1200\,\text{mm} \\
\text{Radius of roof, } r &= \left(\frac{4h^2 + S^2}{8h}\right) \\
&= \left(\frac{5.76 + 36}{9.6}\right) \times 10^3 \\
&= 4350\,\text{mm} \\
\text{Number of lamellas, } n &= 6 \\
\text{Skew angle, } \phi &= 20^0 \\
\text{Thickness of lamella, } t &= 38\,\text{mm} \\
\text{Width of lamella above the bolt, } W &= 150\,\text{mm} \\
\text{Bolt diameter, } b &= 10\,\text{mm}
\end{aligned}$$

From these initial conditions we work out the unadjusted length of a lamella. First, the angle $n\theta$ subtended by half the roof is given by:

$$\begin{aligned}
r \sin n\theta &= S/2 \\
\text{or } \ n\theta &= \arcsin{(S/2r)} = \arcsin{(6000/8700)} \\
6\theta &= 43.6^\circ \\
\theta &= 7.267^\circ \\
\text{so we have, } l &= \left(\frac{2r\sin\theta}{\cos\phi}\right) \\
&= \left(\frac{8700 \times 0.1265}{0.9397}\right) = 1171.1\,\text{mm}
\end{aligned}$$

The unadjusted inter-node distance will be $l \sin\phi = 400.55\,\text{mm}$. The lamella length adjusted for the thickness of the lamella and bolt diameter is:

$$\begin{aligned}
l'' &= l/\cos\phi + t\tan\phi\,(1 + \sec 2\phi) + b(1 + \tan 2\phi \tan\phi) \\
l'' &= 1100.5/0.9397 + 38 \times 0.364 \times (1 + 1.3054) + 6 \times (1 + 0.839 \times 0.364) \\
&= 1216.1\,\text{mm}
\end{aligned}$$

The adjusted inter-node distance is:

$$\begin{aligned}
d' &= d + t(1 + \sec 2\phi)\sec\phi + b\tan 2\phi/cos\phi \\
&= 400.55 + 38 \times (1 + \sec 40^\circ) \times \sec 20^\circ + 10 \times 0.839/0.9397 \\
&= 502.7\,\text{mm}
\end{aligned}$$

Now we adjust the length of the lamella again to account for the amount of the bevel beyond the bolt. The bevel angle ψ is given by:

$$\begin{aligned}
\psi &= \arctan\left(\frac{\sin\phi\cos\phi(1 + \cos\theta)}{1 - \sin^2\phi(1 + \cos\theta)}\right) \\
&= \arctan\left(\frac{\sin 20^\circ \cos 20^\circ (1 + \cos 7.267^\circ)}{1 - \sin^2 20^\circ (1 + \cos 7.267^\circ)}\right) \\
&= \arctan\left(\frac{0.3420 \times 0.9397 \times 1.992}{1 - 0.117 \times 1.992}\right) \\
&= \arctan(0.8347) = 39.85^\circ
\end{aligned}$$

(Note the approximating assumption that the bevel angle is 2ϕ would be good.) The horizontal bevels at both ends thus add $2t/\tan\psi$ or $91.0\,\text{mm}$.

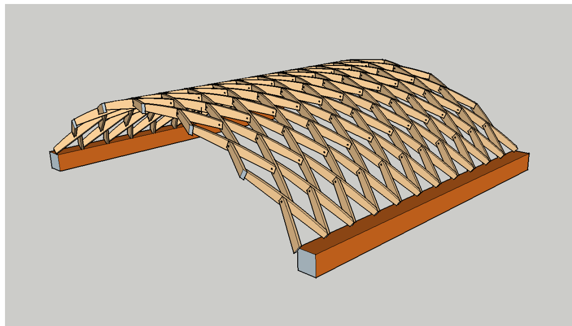

Figure 17: The built-up model

The vertical bevel adds incrementally to this. The angle is given by:

$$
\begin{aligned}
\chi &= \arctan\left(\frac{\tan\theta}{(1/\cos\theta + 1)\cos\phi}\right) \\
&= \arctan\left(\frac{\tan 7.267^\circ}{(1/\cos 7.267^\circ + 1)\cos 20^\circ}\right) \\
&= \arctan\left(\frac{0.1275}{(1/0.992 + 1)\times 0.9397}\right) \\
&= \arctan(0.0676) = 3.87^\circ.
\end{aligned}
$$

Since in this case

$$\theta' = \arctan\left(\frac{\tan\theta}{\cos\phi}\right) = 7.72^\circ$$

the approximation that the vertical bevel angle is $\theta'/2$ is also good. The lamella blank to accommodate the vertical bevel would therefore have an incremental length of $2.W.\tan\chi = 20.3\,\text{mm}$.

The dimensions of the lamella blank are therefore ($1327.4\,\text{mm}\times 150\,\text{mm}\times 38\,\text{mm}$). The built-up roof looks like Figure 17.[4]

References

[1] Allen, J. S. 1999. "A short history of lamella roof construction." *Transactions of the Newcomen Society* Vol. 71 (1).

[4] This is a screenshot of a Sketchup model generated automatically by a Ruby plug-in that uses the geometry described here.

[2] Masani, N. J. 1982. *Theory and Practice of Timber and Lamella Frames.* Delhi: F.R.I. Press

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[5] Timber Engineering Company 1956. *Timber Design and Construction Handbook.* New York: McGraw-Hill.